

# The Dirac Operator Spectrum and Effective Field Theory

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When chiral symmetry is spontaneously broken, the low-energy part of the Dirac operator spectrum can be computed analytically in the chiral limit. The tool is effective field theory or, equivalently in this case, Random Matrix Theory.

## 1 Introduction

Contrary to what one might naively have expected, it is possible to predict the analytical behavior of the Dirac operator spectrum in QCD and QCD-like theories for eigenvalues very close to zero.<sup>a</sup> This is the low-energy regime of the theory, and ordinary QCD perturbation theory is completely irrelevant here. In fact, the analytical predictions turn out to be not only non-perturbative in the QCD coupling constant, but *exact*. These developments began eight years ago with a paper by Leutwyler and Smilga<sup>1</sup>, where for the first time it was shown that if QCD undergoes spontaneous chiral symmetry breaking, there are constraints on the Dirac operator eigenvalues near the origin. So the idea is not new, but there have been many new developments in just the past few months. Here I summarize some of the recent work I have been involved in myself. For a comprehensive review of the subject up to last year, see ref.<sup>2</sup>.

What originally pushed this subject forward was the suggestion by Shuryak and Verbaarschot<sup>3</sup> that the lowest part of the Dirac operator spectrum could be computed analytically by means of Random Matrix Theory. Much work has gone into understanding this remarkable connection between chiral symmetry breaking in gauge theories and Random Matrix Theory. In particular, it has been shown that the chiral (flavor) symmetry breaking in gauge theories coupled to fermions can be classified precisely according to the main categories of Random Matrix Theory ensembles<sup>4,5,6</sup>. In the limit relevant for applications to the Dirac operator spectrum these Random Matrix Theory ensembles actually provide *universality classes*<sup>7</sup>: results do not depend in detail on the particular distributions of the random matrices.

Random Matrix Theory should not be required to derive results for the Dirac operator spectrum; all should follow from field theory alone. It is one of the more recent discoveries that this is indeed the case. There are now three independent means of deriving analytical results for the Dirac operator spectrum:

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- Random Matrix Theory
- The supersymmetric method
- The replica method

Both the so-called supersymmetric<sup>8</sup> and replica<sup>9,10</sup> methods stand out, as they rely only on the field theory formulation itself. Because results are claimed to be *exact*, all three methods should yield identical results. This is indeed the case. One pleasing consequence is that one can now use any of the three different formulations according to what is most convenient. Generically the Random Matrix Theory formulation is the easiest for analytical calculations, but there are cases where it is actually simpler to use the effective field theory partition function<sup>11</sup> (see also refs. <sup>12,13</sup>). Even in those cases, the explicit derivations have usually gone through Random Matrix Theory at intermediate steps.

In this short review I will cover three different topics on which there has been progress in just the last few months. These are: **(i)** the analytical computation of individual eigenvalue distributions<sup>14</sup> (and subsequent lattice measurements<sup>16</sup>), **(ii)** the replica method as applied to the finite-volume effective partition function of QCD<sup>9,10</sup>, and **(iii)** lattice Monte Carlo measurements of the low-energy end of the Dirac operator spectrum near the chiral phase transition<sup>17,18</sup>.

## 2 Distributions of individual Dirac operator eigenvalues

Analytical expressions for the distribution of the first non-zero Dirac operator eigenvalue have been known for some time in the case of all three chiral ensembles. Recently a much more general analytical expression was found for the probability distribution of the  $k$ th smallest Dirac operator eigenvalue (again counted above the exact zero modes)<sup>14</sup>. The formula is completely general, holds for any number of (possibly massive) fermions, and for any of the three major chiral universality classes, as labelled by their Dyson index  $\beta$ . The only technical restriction is that for the universality class of  $\beta = 1$  (corresponding to gauge group SU(2) and fermions in the fundamental representation), the formula only works for a sector of *odd* topological charge  $\nu$ . While the analytical expression has been derived in the framework of Random Matrix Theory, the final formula turns out to depend only on the effective field theory partition functions. It generalizes a related formula for just the lowest eigenvalue distribution in the case of the  $\beta = 2$  universality class, which also expressed the distribution in terms of finite volume partition functions<sup>15</sup>.

Before presenting the resulting closed analytical expression for the distribution of the  $k$ th smallest Dirac operator eigenvalue, let us first recall some basic facts about the connection to Random Matrix Theory. Because chiral symmetry is presumed spontaneously broken, the large-volume euclidean partition function is, in the chiral limit, entirely dominated by the pseudo-Goldstone bosons. The dominance is very strong: contributions from all other physical excitations are exponentially suppressed in the masses, with a coefficient in the exponent that grows linearly with the size of the box. As the size of the box is sent to infinity, the Goldstone bosons dominate the euclidean partition function entirely. It is in this sense that the results for the Dirac operator eigenvalues are exact: they can be made to reach any desired accuracy by taking the volume large enough, and the quark masses small enough. By imposing on the four-volume that  $V \gg m_\pi^{-4}$  only the zero mode piece of the Goldstone field  $\pi(x)$  survives, and the effective partition function, in a sector of topological charge  $\nu$  becomes<sup>1</sup>

$$\mathcal{Z}_\nu^{(N_f)}(\{\mu_i\}) = \int_{U(N_f)} dU (\det U)^\nu \exp \left[ \frac{1}{2} V \Sigma \text{Tr}(\mathcal{M}U^\dagger + U\mathcal{M}^\dagger) \right] . \quad (1)$$

This partition function, a zero-dimensional group integral has, surprisingly, a Random Matrix Theory representation<sup>3</sup>. To get it, one starts with<sup>3</sup>

$$\tilde{\mathcal{Z}}_\nu^{(N_f)}(\{m_i\}) = \int dW \prod_{f=1}^{N_f} \det(iM + m_f) \exp \left[ -\frac{N}{2} \text{tr} V(M^2) \right] , \quad (2)$$

where

$$M = \begin{pmatrix} 0 & W^\dagger \\ W & 0 \end{pmatrix} . \quad (3)$$

The complex matrix  $W$  in eq. (2) is of size  $(N + \nu) \times N$ , and the potential is essentially not constrained beyond yielding a non-vanishing spectral density at the origin,  $\rho(0) \neq 0$ <sup>7</sup>. The limit  $N \rightarrow \infty$  is taken in the Random Matrix Theory partition function in such a way that the combinations  $\mu_i \equiv m_i \pi \rho(0) 2N$ , are kept fixed. In this limit the partition functions  $\mathcal{Z}_\nu^{(N_f)}$  and  $\tilde{\mathcal{Z}}_\nu^{(N_f)}$  become equal up to a  $\mu_i$ -independent constant<sup>3</sup>.

For practical computations an eigenvalue representation of the above matrix integral is convenient. Taking a general Dyson index  $\beta = 1, 2$  or  $4$ , it can, up to an overall and irrelevant normalization factor, be written as

$$Z_\nu^{(\beta)} = \prod_{i=1}^{N_f} m_i^\nu \int_0^\infty \prod_{i=1}^N \left( dx_i x_i^{\beta(\nu+1)/2-1} e^{-\beta x_i} \prod_{j=1}^{N_f} (x_i + m_j^2) \right) \prod_{i>j}^N |x_i - x_j|^\beta , \quad (4)$$

where  $x_i$  are the eigenvalues of  $M^2$ . Here the potential has been chosen to be just  $V(x) = x$ . This is precisely permitted because of universality of the final results<sup>7</sup>. Due to symmetry under  $\nu \rightarrow -\nu$  it is convenient to restrict oneself to  $\nu \geq 0$ .

The unnormalized joint probability distribution for all  $N$  eigenvalues is

$$\rho_N^{(\beta)}(x_1, \dots, x_N; \{m_i^2\}) = \prod_{i=1}^N \left( x_i^{\beta(\nu+1)/2-1} e^{-\beta x_i} \prod_{j=1}^{N_f} (x_i + m_j^2) \right) \prod_{i>j}^N |x_i - x_j|^\beta. \quad (5)$$

Similarly, the unnormalized joint probability distribution of the  $k$  smallest eigenvalues  $\{0 \leq x_1 \leq \dots \leq x_{k-1} \leq x_k\}$  is given by

$$\begin{aligned} \Omega_{N,k}^{(\beta)}(x_1, \dots, x_k; \{m_i^2\}) &= \int_{x_k}^{\infty} dx_{k+1} \dots dx_N \rho_N^{(\beta)}(x_1, \dots, x_N; \{m_i^2\}) \\ &= \prod_{i=1}^k \left( x_i^{\beta(\nu+1)/2-1} e^{-\beta x_i} \prod_{j=1}^{N_f} (x_i + m_j^2) \right) \prod_{i>j}^k |x_i - x_j|^\beta \\ &\times \int_{x_k}^{\infty} dx_{k+1} \dots dx_N \prod_{i=k+1}^N \left( x_i^{\beta(\nu+1)/2-1} e^{-\beta x_i} \prod_{j=1}^{N_f} (x_i + m_j^2) \prod_{j=1}^k (x_i - x_j)^\beta \right) \\ &\times \prod_{i>j \geq k+1}^N |x_i - x_j|^\beta, \end{aligned} \quad (6)$$

Shifting  $x_i \rightarrow x_i + x_k$  in the integrand gives

$$\begin{aligned} \Omega_{N,k}^{(\beta)}(x_1, \dots, x_k; \{m_i^2\}) &= e^{-(N-k)\beta x_k} \prod_{i=1}^k \left( x_i^{\beta(\nu+1)/2-1} e^{-\beta x_i} \prod_{j=1}^{N_f} (x_i + m_j^2) \right) \\ &\times \prod_{i>j}^k |x_i - x_j|^\beta \int_0^{\infty} \prod_{i=k+1}^N \left( dx_i e^{-\beta x_i} x_i^\beta (x_i + x_k)^{\beta(\nu+1)/2-1} \prod_{j=1}^{N_f} (x_i + m_j^2 + x_k) \right. \\ &\times \left. \prod_{j=1}^{k-1} (x_i + x_k - x_j)^\beta \right) \prod_{i>j \geq k+1}^N |x_i - x_j|^\beta. \end{aligned} \quad (7)$$

To finally get the probability distributions of the Dirac operator eigenvalues one takes the microscopic limit  $N \rightarrow \infty$  with, in this particular convention,  $\zeta_i = \pi\rho(0)\sqrt{x_i} = \sqrt{8N}x_i$  and  $\mu_j = \pi\rho(0)m_j = \sqrt{8N}m_j$  kept fixed. In this large- $N$  limit the difference between partition functions based on  $N - k$  and  $N$  eigenvalues disappears. One immediately sees that the new terms in the

integrand of (7) can be interpreted as arising from new additional fermion species, with the partition function now being evaluated in a fixed topological sector of effective charge  $\nu = 1 + 2/\beta$ . The only restriction is that for  $\beta = 1$  the topological charge  $\nu$  must be odd (because otherwise the number of additional fermion species will be fractional).

Taking into account the definition (4), this gives:

$$\begin{aligned} \omega_k^{(\beta)}(\zeta_1, \dots, \zeta_k; \{\mu_i\}) &= \lim_{N \rightarrow \infty} \left( \prod_{i=1}^k \frac{|\zeta_i|}{8N} \right) \Omega_{N,k}^{(\beta)} \left( \frac{\zeta_1^2}{8N}, \dots, \frac{\zeta_k^2}{8N}; \left\{ \frac{\mu^2}{8N} \right\} \right) \\ &= C e^{-\beta \zeta_k^2/8} \zeta_k^{\beta \frac{(\nu+1)}{2} - \nu - 1 + \frac{2}{\beta}} \prod_{j=1}^{N_f} (\mu_j^2 + \zeta_k^2)^{\frac{1}{2} - \frac{1}{\beta}} \prod_{i=1}^{k-1} \left( \zeta_i^{\beta(\nu+1)-1} (\zeta_k^2 - \zeta_i^2)^{\frac{\beta}{2}-1} \times \right. \\ &\quad \left. \prod_{j=1}^{N_f} (\zeta_i^2 + m_j^2) \right) \prod_{i>j}^{k-1} (\zeta_i^2 - \zeta_j^2)^\beta \prod_{j=1}^{N_f} \mu_j^\nu \frac{\mathcal{Z}_{1+2/\beta}^{(\beta)}(\{\sqrt{\mu_i^2 + \zeta_k^2}\}, \{\sqrt{\zeta_k^2 - \zeta_i^2}\}, \{\zeta_k\})}{\mathcal{Z}_\nu^{(\beta)}(\{\mu_i\})} \end{aligned} \quad (8)$$

In the partition function in the numerator each of the  $N_f$  fermion masses have been shifted according to  $\mu_i^2 \rightarrow \mu_i^2 + \zeta_k^2$ . There are  $\beta(k-1)$  new masses  $\zeta_k^2 - \zeta_i^2$ ,  $i = 1, \dots, k-1$ , each of them being  $\beta$  times degenerate. Finally there are also  $\beta(\nu+1)/2 - 1$  new fermions, all of mass  $\zeta_k$ . The overall normalization factor  $C$  is fixed by the requirement of probability conservation.

To get the individual eigenvalue distribution of the  $k$ th eigenvalue, one simply integrates out the previous  $k-1$  smaller eigenvalues, *viz.*,

$$p_k^{(\beta)}(\zeta; \{\mu_i\}) = \int_0^\zeta d\zeta_1 \int_{\zeta_1}^\zeta d\zeta_2 \cdots \int_{\zeta_{k-2}}^\zeta d\zeta_{k-1} \omega_k^{(\beta)}(\zeta_1, \dots, \zeta_{k-1}, \zeta; \{\mu_i\}) . \quad (9)$$

The general formulas (8) and (9) may look rather complicated, but they actually simplify considerably in a number of interesting situations. For instance, in the physically most interesting case of QCD (which belongs to the  $\beta = 2$  universality class<sup>4</sup>), we get in a sector of topological charge  $\nu = 0$ :

$$\begin{aligned} \omega_k(\zeta_1, \dots, \zeta_k; \{\mu_i\}) &= C e^{-\zeta_k^2/4} \zeta_k \prod_{i=1}^{k-1} \left( \zeta_i \prod_{j=1}^{N_f} (\zeta_i^2 + \mu_j^2) \right) \prod_{i>j}^{k-1} (\zeta_i^2 - \zeta_j^2)^2 \times \\ &\quad \frac{\mathcal{Z}_2 \left( \left\{ \sqrt{\mu_i^2 + \zeta_k^2} \right\}, \sqrt{\zeta_k^2 - \zeta_1^2}, \sqrt{\zeta_k^2 - \zeta_1^2}, \dots, \sqrt{\zeta_k^2 - \zeta_{k-1}^2}, \sqrt{\zeta_k^2 - \zeta_{k-1}^2} \right)}{\mathcal{Z}_0(\{\mu\})} \end{aligned} \quad (10)$$

The finite-volume partition functions involved here are known in closed analytical form<sup>19</sup>,

$$\mathcal{Z}_\nu(\{\mu_i\}) = \det A(\{\mu\}) / \Delta(\mu^2) , \quad (11)$$

where the determinant is taken over the  $N_f \times N_f$  matrix

$$A(\{\mu\}) = \mu_i^{j-1} I_{\nu+j-1}(\mu_i) , \quad (12)$$

and

$$\Delta(\mu^2) = \prod_{i>j}^{N_f} (\mu_i^2 - \mu_j^2) . \quad (13)$$

With this convention the normalization factor is  $C = 1/2$  for all values of  $k$ ,  $N_f$  and  $\nu$ .

The analytical formula (10) has very recently been tested by lattice gauge theory simulations<sup>16</sup>, with quite remarkable agreement even at relatively small volumes. These simulations were done with staggered fermions, which are almost totally insensitive to gauge field topology at the couplings we are concerned with here<sup>20</sup>. This means that comparisons should be done only with the  $\nu = 0$  analytical predictions. Shown in figure 1 is the result of a large-statistics computation in quenched SU(3) gauge theory. The individual Dirac operator eigenvalues indeed do have distributions that fall right on top of the analytical predictions.

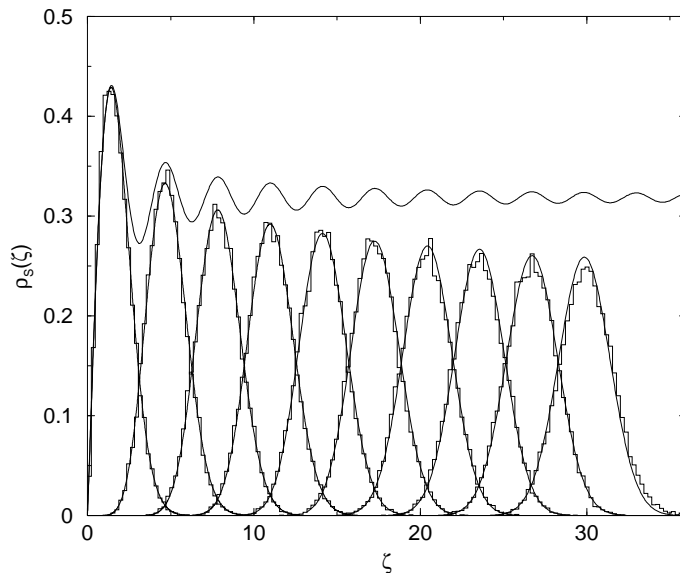


Fig. 1 Individual Dirac operator eigenvalue distributions from lattice Monte Carlo simulations in quenched SU(3) gauge theory<sup>16</sup>, and the analytical predictions of eq. (10).

Simulations with dynamical fermions in SU(3) gauge theory display the same degree of accuracy (although for obvious reasons the statistics here is much lower)<sup>16</sup>. Recall that both masses  $m_i$  and eigenvalues are rescaled with the same factor of  $\Sigma V$ . This means that to get as good accuracy as possible one needs to probe the theory of nearly massless quarks. In figure 2 is shown an analogous plot for the theory with  $N_f = 1$  (strong coupling) staggered fermions (at a mass value of  $m = 0.003$  and a volume of  $6^4$ ).

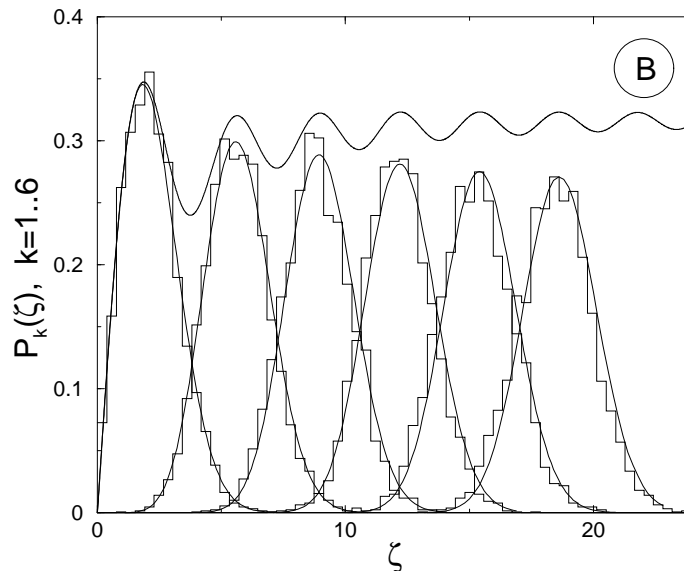


Fig. 2 Individual Dirac operator eigenvalue distributions from lattice Monte Carlo simulations of SU(3) gauge theory with  $N_f = 1$ <sup>16</sup>, compared with the analytical predictions of eq. (10).

One sees the same kind of detailed agreement as in the quenched theory. This establishes quite clearly that not only can the microscopic spectral density of the Dirac operator be computed analytically in the relevant scaling window, also individual Dirac operator eigenvalues are falling right on the analytical predictions.

### 3 The replica method

In lattice gauge theory simulations one often starts out with quenched calculations: the fermion determinant is entirely ignored in the averages. This is

just a simple (sometimes accurate, sometimes not) approximation, introduced only in order to save computer time. A more sophisticated approach is “partial quenching”. Here one does simulate the full theory with dynamical fermions, but in addition one extracts *more* information from the same configurations. This is done by computing quenched averages in the field configurations that already included the effects of dynamical fermions. At first sight it may sound absurd to do this, but in fact there is much genuine physics to be extracted from such partially quenched simulations. In addition, it turns out that at the analytical level partial quenching is just what is needed in order to derive properties of the Dirac operator spectrum. In this case partial quenching is not at all considered as an approximation to the full theory; rather, it is used to obtain genuine physical information about the real theory, the one with propagating dynamical quarks.

To make a partially quenched average, we must measure some correlation function of new fictitious “valence” quarks that do not affect the configurations. The perhaps most simple quantity is the partially quenched chiral condensate. We can get this condensate by adding  $N_v$  valence quarks, differentiate with respect to their mass, and subsequently taking the limit  $N_v \rightarrow 0$ :

$$\begin{aligned} \frac{\Sigma_\nu(\mu_v, \{\mu_i\})}{\Sigma} &\equiv \lim_{N_v \rightarrow 0} \frac{1}{N_v} \frac{\partial}{\partial \mu_v} \ln \mathcal{Z}_\nu^{(N_f+N_v)}(\mu_v, \{\mu_i\}) \\ &= \frac{1}{\mathcal{Z}_\nu^{(N_f+N_v)}(\mu_v, \{\mu_i\})} \frac{\partial}{\partial N_v} \frac{\partial}{\partial \mu} \mathcal{Z}_\nu^{(N_f+N_v)}(\mu_v, \{\mu_i\}) \Big|_{N_v=0} \end{aligned} \quad (14)$$

In the last line we have formally expanded the partition function as a Taylor series in  $N_v$ . It is not obvious that precise meaning can be given to such a notion, since the function involved *a priori* is known only at integer values of  $N_v$ .

This particular way of deriving the partially quenched chiral condensate is known as the replica method. Its applicability in certain condensed matter contexts has been questioned<sup>21</sup>, but last year there was considerable progress towards understanding how to apply this method to derive spectral properties<sup>22</sup>. In the present context of QCD, the partially quenched chiral condensate in the same finite-volume scaling regime as discussed above was computed in refs.<sup>9,10</sup>. The replica method trivially works in ordinary QCD perturbation theory (it simply kills all closed fermion loops, as required). It is a more non-trivial fact that the replica method also is suited for deriving quantities that are non-perturbative in the QCD coupling constant. For instance, in the low-energy framework of effective chiral Lagrangians the replica method<sup>23</sup> works quite analogous to the previously known supersymmetric method<sup>24</sup>.



Although in principle the partially quenched chiral condensate is an unphysical quantity, it turns out that this is not *quite* so. There is one tiny bit of this quantity that contains important physics: this is the discontinuity across the imaginary mass axis<sup>8</sup>, which gives the spectral density. In the microscopic scaling region:

$$\begin{aligned}\rho_S^{(\nu)}(\zeta; \{\mu_i\}) &= \frac{1}{2\pi} \text{Disc } \Sigma_\nu(\mu_v, \{\mu_i\})|_{\mu_v=i\zeta} \\ &= \frac{1}{2\pi} [\Sigma_\nu(i\zeta + \epsilon; \{\mu_i\}) - \Sigma(i\zeta - \epsilon; \{\mu_i\})] .\end{aligned}\quad (15)$$

In ref. <sup>9</sup> two expansions of the partially quenched chiral condensate were considered: small-mass and large-mass expansions. It turned out that neither were suitable for deriving the spectral density of the Dirac operator. The small-mass expansion suffered from so-called de Wit–t Hooft poles<sup>25</sup>, while the large-mass expansion of ref. <sup>9</sup>, based as it were on the leading saddle point, only gave the exponentially leading asymptotic series near the real axis; it could not be trusted near the imaginary axis, as required to get the spectral density. Quite recently, Dalmazi and Verbaarschot<sup>10</sup> have shown how to repair this latter deficiency of the large-mass expansion, by including other saddles. Indeed, their asymptotic expansion for large masses is valid also near the imaginary axis. Taking the discontinuity there according to eq. (15) precisely reproduces the asymptotic expansion of the microscopic spectral density of the Dirac operator as computed based on the Random Matrix Theory formulation. This is a highly non-trivial fact, since in detail the computations are entirely different from those of either the Random Matrix Theory context or the supersymmetric formulation. Conceptually, the replica method has the advantage that the pattern of chiral symmetry breaking safely can be assumed to be the conventional one:  $\text{SU}(N_f + N_v)_L \times \text{SU}(N_f + N_v)_R \rightarrow \text{SU}(N_f + N_v)$ , whereas in the supersymmetric formulation the symmetry breaking pattern is simply assumed to mimic as closely as possible the known bosonic one. The corresponding difficulty in the replica method is obviously how this pattern of chiral symmetry breaking can be given meaning for non-integer  $N_v$  (or rather, infinitesimal  $N_v$ , which is all that is required). The fact that the replica method happens to agree with the supersymmetric one can, since they are so entirely different, be seen as an independent confirmation of both.

#### 4 Smallest Dirac operator eigenvalues near $T_c$

Because of the Banks-Casher relation between the infinite-volume chiral condensate  $\Sigma$  and the spectral density of the Dirac operator at the origin,  $\Sigma = \pi\rho(0)$ , it is obviously of interest to trace the depletion of Dirac eigenvalues at

finite temperature. A lattice study of this question was first performed in ref. <sup>17</sup>, and this spring similar issues were addressed with higher statistics <sup>18</sup>.

The obvious question to ask is: can the Random Matrix Theory formulation of the effective Lagrangian (1) be used to predict the Dirac operator spectrum near  $T_c$ ? The answer is clearly in the negative, since the effective finite-volume chiral Lagrangian (1) simply is incorrect at finite temperature. The best one can do is to take into account the leading effect of replacing the symmetric euclidean four-volume  $V = L^4$  by an asymmetric one,  $V = L^3/T$ , where  $T$  still is on the order of  $1/L$ . This means that this effective chiral Lagrangian is suitable only for probing infinitesimally small temperatures, and the same goes for the associated Random Matrix Theory formulation.

This does not mean that it is uninteresting to study the behavior of the smallest Dirac operator eigenvalues as the temperature is increased towards the critical temperature  $T_c$  of the chiral phase transition. For instance, one observation of ref. <sup>18</sup> was that tracing the evolution in the magnitude of just the single smallest Dirac operator eigenvalue is a remarkably simple way to get an accurate determination of the phase transition point. Shown in figure 3 is an example of this, taken from ref. <sup>18</sup>.

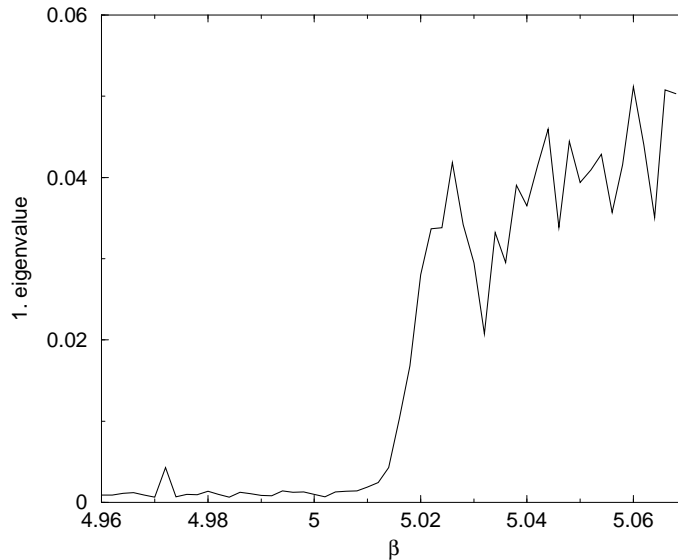


Fig. 3 Just one single Dirac operator eigenvalue, traced through a series of ensembles with almost no averaging, suffices for getting an accurate determination of the coupling at which the chiral phase transition occurs. From ref. <sup>18</sup>.

A more ambitious goal would be to try to compute the corresponding microscopic spectral density at, say, precisely the critical temperature  $T_c$ . (For this to make sense, one should need a number of fermions  $N_f$  for which the transition is continuous). The procedure is as clear as in the case of zero temperature, but a systematic way of obtaining the effective theory is lacking. Presumably the closest one can get is an effective Lagrangian framework like that of Pisarski and Wilczek<sup>26</sup>. This concerns, however, an effective theory of which not even the leading behavior as a function of mass is known analytically. Progress on this front is more likely to come solely from the numerical side.

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